THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH2060B Mathematical Analysis II (Spring 2017)
Tutorial 2

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This tutorial is mainly devoted to the application of derivatives.

1. Suppose $f : [a, b] \to \mathbb{R}$ is continuous and is differentiable on (a, b).
   (a) State the mean value theorem.
   (b) (Optional) It should be noted that the condition that f is real valued is necessary. For a counterexample, consider the function $f : [0, 2\pi] \to \mathbb{C}$ defined by $f(\theta) := e^{i\theta}$. Then $f(0) = f(2\pi)$, but there is no $\theta \in (0, 2\pi)$ such that $f'(\theta) = 0$.
   (c) Suppose $f'(c) \geq 0$ for any $c \in (a, b)$. Then f is monotonically increasing on $[a, b]$.
   (d) Conversely, suppose $f'(c) < 0$ for some $c \in (a, b)$. Then we have shown last time that f is not increasing on $[a, b]$.
   (e) Suppose now $f'(c) > 0$ for any $c \in (a, b)$. Then show that f is **strictly** increasing on $[a, b]$.
   (f) However, this time the converse is not true. Provide a counterexample.
   (g) Using differentiation and monotonicity, compare the values of e^π and π^e without a calculator.

2. (a) Show that $|\sin(x) - \sin(y)| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
   (b) Show that $\cos(x) > 1 - \frac{x^2}{2}$ for all $x > 0$.
   (c) Show that for $x > 1$, $\frac{x-1}{x} < \ln x < x - 1$.
   (d) Show that for any $0 < a < b$, there is $c \in (a, b)$ such that $\sqrt{c} = \frac{\sqrt{a}+\sqrt{b}}{2}$.
   (e) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable, and that $|f'(x)| \leq 50$ for all $x \in \mathbb{R}$. Given that $f(1997) = 10000$, compute the following value:

$$\inf\{f(2047) : \text{ all } f\text{'s satisfying the above conditions}\}$$

   and provide an example such that a minimum can be attained.

3. Let $f : [a, b] \to \mathbb{R}$ be continuous and is differentiable on (a, b).
   (a) Show that if f attains a local maximum (minimum) at $c \in (a, b)$, then we must have $f'(c) = 0$. However, give an example to show that the converse is false.
   (b) Let a, b be two positive numbers and $p \geq 1$ be a real number. Show that

$$a^p + b^p \leq (a + b)^p \leq 2^{p-1}(a^p + b^p)$$

(c) (Young's inequality) Let a, b be two positive numbers and $p, q > 1$ be two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

4. (a) State the L'Hospital's Rule.

(b) Compute the limit $\lim_{x \to 0^+} x^x$.

(c) Note that the converse of L'Hospital's Rule fails in the sense of following example:

Consider $f(x) := x^2 \sin(\frac{1}{x})$ for $x \neq 0$ and $f(0) := 0$, $g(x) := \sin x$. Then $\lim_{x \to 0} f(x) = 0 = \lim_{x \to 0} g(x) = 0$ so this is an indeterminate form. Note that

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = 0,$$

but this limit cannot be evaluated using L'Hospital Rule, since we have

$$\lim_{x \to 0} \frac{f'(x)}{g'(x)} \text{ does not exist}$$

(d) Show by L'Hospital Rule that whenever $f''(x)$ exists and is continuous in $\mathbb{R}$, we have

$$\lim_{t \to 0} \frac{f(x+t) + f(x-t) - 2f(x)}{t^2} = f''(x)$$

This is useful because it provides us a way to represent higher order derivatives with no reference to intermediate derivatives.

Give an example such that the limit on the left and side exists but f is not even continuous at x.

Remarks: Let a, b be two positive numbers and $p, q > 1$ be two real numbers satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Show that

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}$$

Solution:

An equivalent statement to the one we want to prove is the following:

$$\frac{a}{b^{q-1}} \le \frac{1}{p} \cdot \frac{a^p}{b^q} + \frac{1}{q}$$

(divide both sides by b^q).

Let $t = \frac{a^p}{b^q}$. Then $t^{\frac{1}{p}} = \frac{a}{b^{q-1}}$, where we have used the assumption that $\frac{1}{p} + \frac{1}{q} = 1$.

We want to show that

$$t^{\frac{1}{p}} - \frac{1}{p} t \le \frac{1}{q}$$

Let $f(t) = t^{\frac{1}{p}} - \frac{1}{p}t$. Then $f(1) = 1 - \frac{1}{p} = \frac{1}{q}$. If we can show that this is the maximum of $f(t)$ when $t \geq 0$, we are done.

$$f'(t) = \frac{1}{p}t^{\frac{1}{p}-1} - \frac{1}{p} = \frac{1}{p}\left(t^{-\frac{1}{q}} - 1\right)$$

When $t \leq 1$ we see that $f'(t) \geq 0$ and similarly when $t \geq 1$, we see that $f'(t) \leq 0$. Thus, $\frac{1}{q}$ is a local max of $f(t)$ attained at $t = 1$. However, this is the unique critical point of f and we see that f decreases when t moves away from 1. Hence f attains a global maximum at $t = 1$, and hence

$$t^{\frac{1}{p}} - \frac{1}{p}t \leq \frac{1}{q}$$

Now substituting back in $t = \frac{a^p}{b^q}$ we get Young's Inequality to come out with some algebraic manipulations.